Multivariate Linear Regression

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Simple Linear Regression - Least Squares

Find best linear function $f(x) = \omega_0 + \omega_1 x$

Oservations (x_i, y_i) Regression Coef ω_0 & ω_1

Residual Sum of Squares

$$\text{SSE} = \sum_{i=1}^{N} [y_i - f(x_i)]^2 = \sum_{i=1}^{N} [y_i - \omega_1 x_i - \omega_0]^2$$

$$\frac{\partial \mathbf{E}}{\partial \omega_0} = -2 \sum_{i=1}^{N} [y_i - \omega_1 x_i - \omega_0] = 0$$

$$\frac{\partial \mathbf{E}}{\partial \omega_1} = -2 \sum_{i=1}^{N} [y_i - \omega_1 x_i - \omega_0] x_i = 0$$

Normal Equation

$$\frac{\partial \mathbf{E}}{\partial \omega_0} = -2\sum_{i=1}^{N}[y_i - \omega_1 x_i - \omega_0] = 0$$

$$\frac{\partial \mathbf{E}}{\partial \omega_1} = -2\sum_{i=1}^{N}[y_i - \omega_1 x_i - \omega_0]x_i = 0$$

$$\begin{bmatrix} N & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} \sum_i y_i \\ \sum_i x_i y_i \end{bmatrix}$$

Normal Equation Solution

$$f(x) = \overline{y} + \frac{\sigma_{xy}}{\sigma_{xx}} [x - \overline{x}]$$

where
$\sigma_{xy} = \sum_i (x_i - \overline{x})(y_i - \overline{y})$,
$\sigma_{xx} = \sum_i (x_i - \overline{x})^2$,
$\overline{x}$, and $\overline{y}$ are averages.

Multivariate Linear Regression

First Matrix Notation:

Let $\mathbf{X} = (1\ \mathbf{x})$ where

$1 = (1, 1, 1, ..., 1)^T$, $\mathbf{x} = (x_1, x_2, ..., x_N)^T$,

and $\mathbf{y} = (y_1, y_2, ..., y_N)^T$

Multivariate Normal Equation

left-hand side matrix

$\mathbf{X}^T\mathbf{X} =$

$$\begin{bmatrix} 1^T1 & 1^T\mathbf{x} \\ \mathbf{x}^T1 & \mathbf{x}^T\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{N} & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix}$$

Multivariate Normal Equation

right-hand side matrix

$$[\mathbf{1} \quad \mathbf{x}]^T \, \mathbf{y} =$$

$$\begin{bmatrix} \mathbf{1}^T \mathbf{y} \\ \mathbf{x}^T \mathbf{y} \end{bmatrix} = \begin{bmatrix} \sum_i y_i \\ \sum_i x_i y_i \end{bmatrix}$$

Multivariate Normal Equation

Substituting the right hand matrix and left hand matrix into the Normal Equation results in

$\mathbf{X}^T\mathbf{X}\Omega = \mathbf{X}^T\mathbf{y}$ where $\Omega = (\omega_0, \omega_1)^T$

$$\Omega = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Multivariate Linear Regression

Attribute set consists of d explanatory attributes $(x_1, x_2, ..., x_d)$, $\mathbf{X}$ becomes an $N \times d$ design matrix

$\mathbf{X} =$

$$\begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1d} \\ 1 & x_{21} & x_{22} & \cdots & x_{2d} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{N1} & x_{N2} & \cdots & x_{Nd} \end{bmatrix}$$

$\Omega = (\omega_0, \omega_1, ..., \omega_{d-1})^T$ is a d-dimensional vector